Endogenous trading in Credit Default Swaps

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ENDOGENOUS TRADING IN CREDIT DEFAULT SWAPS

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ABSTRACT. We introduce a real options model in order to quantify the moral hazard impact of credit default swap (CDS) positions on the corporate default probabilities. Moral hazard is widely addressed in the insurance literature, where the insured agent may become less cautious about preventing the risk from occurring. Importantly, with CDS the moral hazard problem may be magnified since one can buy multiple protections for the same bond.

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1. INTRODUCTION

Recently there has been much debate about the alleged role of the credit default swaps (CDS) in the US credit crisis during 2007 and 2008 as well as in the European sovereign crisis. A CDS is a negotiated financial contract between a seller and a buyer written on a reference entity's bond over a fixed period of time. The buyer makes regular premium payments to the seller and the seller of the contract guarantees the buyer protection in case

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of a credit event of the reference entity (typically default, bankruptcy or some other types of financial restructuring). Therefore, CDS are similar to classical insurance contracts, since they provide insurance against default. However, one important difference between insurance contracts and CDS is that CDS can be used for speculative purposes. Indeed, an investor can enter a CDS contract without holding the reference entity's bonds. In this case, the investor is not exposed to the risk and the CDS contract is said to be naked. However, in insurance, the principle of insurable interest holds and contracts of this type are not allowed. The research papers by Jarrow [12], Stulz [21], Brunnermeier [4] provide a detailed analysis of the CDS markets, their potential costs and benefits for the economy, including during debt crisis.

The focus in this paper is on the problem of moral hazard. This issue has been well studied in the literature of insurance and it addresses the problem that the insured agent may become less cautious about preventing the risk from occurring. Hence, the insurance may result in an increased likelihood of the insured event. The moral hazard problem can be magnified in the presence of big CDS positions when they represent overinsurance against default. In this paper we propose a model, where the management decisions of the company can be influenced by a shareholder who holds CDS written on the company. The issue is to investigate to what extent this economic agent will influence the investment strategy of the firm with the objective to increase the probability of default. We quantify the impact on the firm's value and firm's default probability (microeconomic impact). We do not investigate any macroeconomic impacts, i.e., effects of CDS on the whole economy. Also, we do not propose an equilibrium model in the sense that there is a liquid market for CDS and the investor can buy a limited quantity of CDS without having a market impact.

We consider a firm that holds an investment and has a real option to switch to another investment opportunity, which is initially suboptimal with respect to the first investment. The switching decision is irreversible. The firm has liabilities to reimburse at a finite maturity date. The default event occurs in case the firm has insufficient cash to pay these liabilities at the given maturity date. In this framework, an investor holds a proportion of the overall firm's liabilities (stocks and debt). This enables him/her to influence the management decisions of the firm, including the switching time determined by the company to the initially suboptimal investment. Furthermore, this investor has the possibility to buy CDS written on the firm. These CDS represent a bet on the default of the firm and thus may lead to moral hazard. In particular, a possible strategy of the agent is to spend as much as possible to buy CDS contracts and enhance the chances of default by exercising the switching option with a loss.

To analyze the above outlined model, we first consider the market value of the firm when the investor is not allowed to trade in CDS. This is done by computing the optimal switching time to the initially suboptimal investment. Then, we formulate an optimization problem, when the investor can invest initially some proportion of the available cash for CDS contracts. Hence, the investor's strategy constitutes of two elements: First the initial allocation of cash and CDS contracts and second the switching time. However, for this analysis, we need the initial market price of the CDS contracts. By assuming that the market is opaque, we note that the CDS position of the investor is not observed by market participants at time 0. Thus, the CDS is priced as if the optimal switching time remains the same, although the investor is able to influence it. In this setup, we investigate how the firm's value with trading CDS is modified.

This choice of strategies for the investor lead us to work within a time-dependent non-standard optimal stopping framework. After transforming the initial problem into a time-dependent classical optimal problem we run numerical simulations to study it, using the Longstaff-Schwartz algorithm. In our numerical experiments we compare the respective firm's values with and without the possibility of investing in CDS. Moreover, we study default probabilities of the firm associated with different strategies. We investigate the impact of different parameters on the firm values and default probabilities. The different parameters we consider are the initial cash available, the debt level, the volatilities and the means of the investments, their correlation, as well as the interest rate.

We observe that there are situations where the investor chooses to exercise the switching option with a loss. This strategy will enable him to substantially increase the default probability of the firm so that the optimal value with CDS exceeds the value in the absence of CDS trading. These situations generally correspond to large positions in CDS with several times the complete insurance. Since the investor chooses to exercise the switching option with a loss, it sheds light on the moral hazard problem. However, we also observe indifference zones, where it is hard to distinguish where CDS trading is actually profitable for the investor. In these zones, CDS positions are relatively small so that the investor does not yield additional gains.

The problem of moral hazard in relation with the CDSs has received increasing attention in the literature recently. The empirical study by Subrahmanyam et al. [22] shows that the credit risk of reference firms, reflected in rating downgrades and bankruptcies, increases significantly upon the inception of CDS trading. Hu and Black ([10],[11]) and Hu [9] introduced the term of empty creditor to name the hedged creditor. They pointed out that investors that hold bonds for a company near bankruptcy and CDS hedges for those bonds have interest to chose the bankruptcy option, triggering a "credit event" and receive the full-payment on bonds in cash, instead of negotiating with the company in the restructuring process. This behavior increases the default probabilities of firms. A series of models show that the empty creditor problem is more complex (Arping [2], Goderis and Wagner [8] and Sambalaibat [20] who use sovereign debt models, Bolton and Oehmke [3]): hedged creditors have indeed an increased bargaining power in ex post debt renegotiations but this can be partially balanced by a disciplining effect ex ante which prevents excessive risk taking by shareholders, hence reduce the default probabilities.

All the above mentioned literature is concerned with describing how the lending relationship is affected by the creditors owning CDSs. Our aim here is of a different nature. We are attached to study the optimal investment decisions in presence of the CDSs as compared to the situation without the CDSs, when an investor is able to exercise influence in one firm's decisions. In this setting, we show how much the firm's value and default probability are affected, depending on the size of the CDS position. Precisely because the relationship debtor-creditor is not within the scope of this paper, we assume that the investor holds a proportion of the overall firm's liabilities (i.e., stocks and debt) and a CDS position, we thus avoid asymmetric incentives induced when only particular stakeholders of the firm hold CDSs.

The remaining of the paper is structured as follows. The model is introduced in Section 2; Section 3 determines the optimal investor's strategies in absence of CDS. In Section 4, we quantify the default probability of the firm for the different scenarios of exercise of the exchange option. Section 5 introduces the optimization problem when the investors has the possibility of investing a given amount of cash in CDS while holding the firm's liabilities. We then present some numerical illustrations in Section 6. Section 7 concludes the paper.

2. A model of a firm with an option to switch the assets

We consider a continuous-time model, where a firm holds an investment and considers at any time the option to replace it with an investment opportunity (i.e., an asset replacement, or switching option). We denote by $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space, where $\mathbf{P}$ is supposed to be a pricing measure for the financial investors in the economy. W^1 and W^2 are correlated Brownian motions under the pricing measure $\mathbf{P}$ with $\langle W^1, W^2\rangle_t = \rho t$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by W^1 and W^2.

We assume that the firm's current investment (a project or an asset) generates net cash flows at time $t \geq 0$ described by:

$$S_t^1 = x + \mu_1 t + \sigma_1 W_t^1, \quad S_0^1 = x > 0.$$

The firm also has an investment opportunity (a second project or asset) that generates net cash flows as:

$$S_t^2 = y + \mu_2 t + \sigma_2 W_t^2 \quad S_0^2 = y > 0, \quad y < x.$$

Also, we assume that the default-free short rate is constant, denoted by r, and $\mu_1 > \mu_2$. As an example, in a duopoly, the firm's investment opportunity consists in the possibility of using the same technology as the competitor, hence S^2 is the net cash flows of one firm when the two competitor firms use the same technology. Here we suppose that at time zero, the firm has a competitive advantage that is, it uses a more profitable technology (since $x > y$). In the future the competitive advantage may disappear (if $S^1 < S^2$) and the firm has the option to swap to the same technology as the competitor firm. This asset replacement is a continuous-time and irreversible decision to be taken, in the sense of McDonald and Siegel [17].

The NPV (net present value) of the first asset, measured as the present value of the future cash flows is given by:

$$\begin{aligned} v_t^1 :=& \mathbf{E}\left[\int_t^\infty e^{-r(u-t)} S_u^1 du | \mathcal{F}_t\right] = \int_t^\infty e^{-r(u-t)} \mathbf{E}\left[S_u^1 | \mathcal{F}_t\right] du \\ =& \int_t^\infty e^{-r(u-t)} (S_t^1 + \mu_1(t-u)) du = \frac{S_t^1 + \mu_1/r}{r}. \end{aligned}$$

Similarly, for the second asset the NPV is:

$$v_t^2 := \mathbf{E}\left[\int_t^\infty e^{-r(u-t)} S_u^2 du | \mathcal{F}_t\right] = \frac{S_t^2 + \mu_2/r}{r}$$

In this context, at time 0 the first project dominates the second one since its NPV is higher. However, we will show later on that in the presence of CDS, the second asset might be preferred.

At time 0 the firm holds the first asset, with NPV v_0^1 and a real option to operate in the future a switch from the first asset to the second one (the asset replacement problem). We assume that the firm has as a liability a debt contract, for simplicity zero coupon bonds, with maturity date T and total nominal value N. In view of being able to reimburse the debt when due, we assume that the net cash flows generated by the firm's assets are capitalized on a risk-free account, until the time $T > 0$.

More precisely, let us denote:

$$a_t^1 := \int_0^t e^{r(t-u)} S_u^1 du$$
$$a_t^2 := \int_0^t e^{r(t-u)} S_u^2 du$$

the cumulated net cash flow processes associated with each asset. Suppose that the firm intends to switch the first asset against the second one at some stopping time θ. We denote by (A_t^θ) the cumulated net cash flow process generated by the firm's activity until time t and associated with a switching time θ, i.e.:

$$\begin{aligned} A_t^\theta &:= \int_0^{t\wedge\theta} e^{r(t-u)} S_u^1 du + \int_{t\wedge\theta}^t e^{r(t-u)} S_u^2 du \\ &= \int_0^t e^{r(t-u)} (S_u^1 \mathbf{1}_{\{\theta>u\}} + S_u^2 \mathbf{1}_{\{\theta\leq u\}}) du = a_{t\wedge\theta}^1 + \mathbf{1}_{\{\theta\leq t\}} \left(a_t^2 - a_\theta^2\right). \end{aligned}$$

A^θ can be negative: indeed, when the net cash flows are negative for a while, the cumulated cash account may become negative. The implicit assumption is that the firm has a line of credit to draw from when the cash account is empty and the net cash flows from operating the business are negative. The cumulated cash (when A^θ is positive) and the credit line (when A^θ is negative) allow the firm to continue operating in periods of negative cash flows. We assume the line of credit is time-limited and has the same maturity date T as the bonds, meaning that at time T the firm has to reimburse not only the bonds, but also the credit line, if used[1].

We assume that the firm is in the default state if at the maturity of the debt contract T, the cash available is insufficient to reimburse the nominal value of the debt, that is, if:

$$A_T^\theta < N.$$

Let us denote by $\mathcal{T}_t$ the class of stopping times θ with $\theta \geq t$, that represent possible switching times of the two assets, after time t. For a given switching time $\theta \in \mathcal{T}_0$, the firm's

[1] Note that for simplicity we use for the line of credit the risk-free interest rate, which is lower than the firm's cost of borrowing. It can be assumed that the extra yield that normally creditors ask for holding the firm's risky debt is included either in an initial fee the firm pays to access this credit facility, or in the final payment N. However, it is not important in our model to specify this issue.

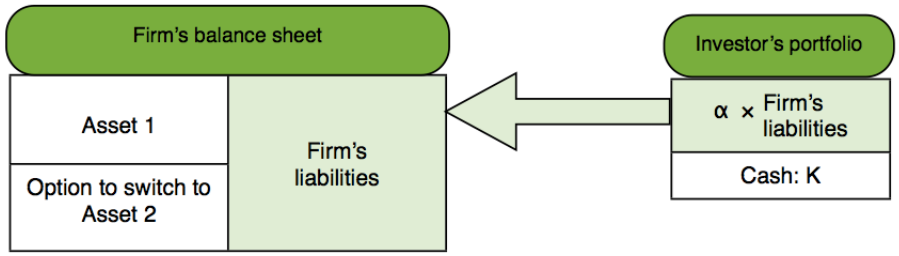

FIGURE 1. Time 0: the investor holds the firm's liabilities and cash in amount K.

economic value at time $t \geq 0$ is:

$$V_t^\theta = (v_t^1 + \mathbf{E}\left[e^{-r(\theta-t)}(v_\theta^2 - v_\theta^1)|\mathcal{F}_t\right])\mathbf{1}_{\theta>t} + v_t^2\mathbf{1}_{\theta\leq t} + A_t^\theta. \tag{1}$$

As long as the firm has not yet operate the switching, the firm's value can be maximized by switching optimally the first asset against the second one:

$$V_t := ess\sup_{\theta\in\mathcal{T}_t} V_t^\theta = (v_t^1 + a_t^1) + ess\sup_{\theta\in\mathcal{T}_t} \mathbf{E}\left[e^{-r(\theta-t)}(v_\theta^2 - v_\theta^1)|\mathcal{F}_t\right]. \tag{2}$$

The solution of this problem will be detailed in Section 3, where we show in particular that there exists an optimal switching time T_ℓ such that $V_0 = v_0^1 + \mathbf{E}\left[e^{-rT_\ell}(v_{T_\ell}^2 - v_{T_\ell}^1)\right]$.

We further suppose that at time 0, an investor holds a proportion $\alpha \in [0,1]$ of the overall firm's liabilities and cash in amount $K \geq 0$. Moreover, we assume that the investor controls the investment decisions (for instance the investor is a controlling shareholder) of the firm and in particular can decide the switching time of the assets. The investor's portfolio in such a situation is illustrated in Figure 1. The investor's portfolio attains a maximal value of $\alpha V_0 + K$, provided the firm's value is also maximized to V_0, by choosing to switch the two assets time T_ℓ. In other words, the optimal firm's strategy (switching at T_ℓ) is also optimal for the investor in this case.

Suppose now that at time 0 the investor decides to invest part of the cash in a portfolio of CDS. Is it easy to see that the investor's economic interest is different in this case, and there are incentives to deviate from the optimal firm's strategies. In other words, it is possible to increase the value of the portfolio by investing in CDS and operating a switching of the assets at times different from T_ℓ. Thus, holding CDS is not neutral for the resulting firm's policies. Our aim here will be to see how badly the optimal strategies are affected, depending on the size of the CDS position.

For computational convenience, we shall suppose that the investor can buy some simple CDS contracts that pay one dollar in case of default of the firm and zero otherwise (this is a reasonable assumption for instance when the investor buys CDSs linked to the lowest seniority debt of the firm). We assume that the market investors anticipate that the optimal switching time is T_ℓ, since this switching time maximizes the firm's value, therefore the price of one CDS time 0 is:

$$C_0 = e^{-rT}\mathbf{P}(A_T^{T_\ell} < N).$$

If the investor decides to invest at time 0 a proportion π of the available cash K in CDS, the cash amount used time 0 for investment in CDS equals πK, that is, the investor buys

$$n(\pi) := \frac{\pi K}{C_0}$$

CDS contracts. The rest of $(1-\pi)K$ is kept as cash. However, the time T payment of the CDS depends on the particular switching strategy that will be eventually implemented and which can be different from the anticipated one. Hence, at time T, the value of the investor's portfolio with corresponding strategies (π, θ), with $\theta \in \mathcal{T}_0$ being a switching time from the asset 1 to the asset 2 and $\pi \in [0,1]$ being the proportion of the available cash invested in CDS at time 0, is given by:

$$v_T^{\pi,\theta} = \alpha V_T^\theta + (1-\pi)Ke^{rT} + n(\pi)\mathbf{1}_{A_T^\theta < N}$$

We denote by v^* the maximal value of the investor's portfolio that can be obtained by an optimal combination of CDS investment and switching of the firm's assets:

$$v^* := \sup_{(\theta,\pi)} e^{-rT}\mathbf{E}[v_T^{\pi,\theta}].$$

We shall prove in Section 5 that :

$$v^* \geq \alpha V_0 + K,$$

reflecting the fact that there is a benefit for the investor from investing in CDS. Furthermore, it is always optimal to invest 100% of the available cash K in CDS. The optimal switching strategies and the corresponding profits for the investor are analyzed in Section 6.

3. Optimal firm value and optimal switching time in absence of CDS investment

In this section we give the expression of the strategy that maximizes the value of the firm, that is: $V_0 = \sup_{\theta\in\mathcal{T}_0} V_0^\theta$. Assuming that market investors anticipate that the manager acts to maximize the firm's value, V_0 is the market value of the firm at time 0.

We denote:

$$v_t = v_t^2 - v_t^1, \quad t \geq 0.$$

The process v is also an arithmetic Brownian motion with negative drift, more precisely:

$$v_t = v_0 + mt + \sigma B_t,$$

where $v_0 = (y-x)/r + (\mu_2 - \mu_1)/r^2$, $m = (\mu_2 - \mu_1)/r$, $\sigma = \sqrt{\sigma_1^2/r^2 + \sigma_2^2/r^2 - 2\sigma_1\sigma_2\rho/r^2}$ and $B_t = (\sigma_2 W_t^2 - \sigma_1 W_t^1)/(\sigma r)$.

Notations. We introduce the following stopping time:

$$\tau_c := \inf\{t : v_t = c\}.$$

Furthermore, for any stopping time θ, we denote:

$$f^\theta(u) := \mathbf{E}[e^{-r\theta} v_\theta | v_0 = u].$$

Lemma 3.1 (Value of the switching option)**.** *We have:*

$$f^{\tau_c}(u) = \begin{cases} ce^{a_1(u-c)}, & u \leq c \\ ce^{a_2(u-c)}, & u \geq c, \end{cases} \tag{3}$$

where $a_{1,2} = (-m \pm \sqrt{m^2 + 2\sigma^2 r})/\sigma^2$. *Furthermore:*

$$f(u) := \sup_{\theta \in \mathcal{T}_0} f^\theta(u) = f^{\tau_L}(u),$$

with

$$L = 1/a_1 \vee u.$$

The function $f(u)$ *is the value of the switching option conditional on* $v_0 = u$*, and* τ_L *is the optimal switching time of the two investments.*

Proof. See the appendix. □

For any constant a, we introduce the first hitting time of $(-\infty, a]$ by the process $Z_t := S^1_t - S^2_t$:

$$T_a(t) := \inf\left\{s \geq t \;\middle|\; Z_s := S^1_s - S^2_s \leq a\right\} = \inf\left\{s \geq t \;\middle|\; v_s \geq -\frac{a + (\mu_1 - \mu_2)/r}{r}\right\}.$$

For simplicity, we denote $T_a := T_a(0)$.

Note that in our context $v_0 = (y-x)/r + (\mu_2 - \mu_1)/r^2 < 0$ and $L > 0$, hence, the optimal switching time τ_L characterized in the Lemma 3.1 can be expressed more conveniently as a first hitting time for the process (Z_t). Indeed, we have: $\tau_L = T_\ell > 0$, where

$$\ell := -Lr - (\mu_1 - \mu_2)/r = -\frac{r}{a_1} - \frac{\mu_1 - \mu_2}{r} < 0.$$

Intuitively, the decision of replacing the current investment will be taken when the generated cash flows will become too low as compared to the cash flows corresponding to the investment opportunity.

We assume that market investors anticipate that the switching time of the assets is T_ℓ, since this switching time maximizes the firm's value. It follows from the previous proposition that:

Corollary 3.2. *The market value of the firm at time* 0 *is:*

$$\begin{aligned} V_0 &= \sup_{\theta \in \mathcal{T}_0} V_0^\theta = \sup_{\theta \in \mathcal{T}_0} \mathbf{E}\left[e^{-r\theta}(v^2_\theta - v^1_\theta)\right] + v^1_0 \\ &= v^1_0 + f(v_0), \end{aligned} \tag{4}$$

and the anticipated switching time is:

$$T_\ell = \inf\left\{s \geq 0 \;\middle|\; S^1_s - S^2_s \leq \ell\right\},$$

where:

$$\begin{aligned}
\ell &= -\frac{r}{a_1} - \frac{\mu_1 - \mu_2}{r} < 0, \\
a_1 &= (-m + \sqrt{m^2 + 2\sigma^2 r})/\sigma^2 \\
m &= -(\mu_1 - \mu_2)/r.
\end{aligned}$$

4. An analysis of the default probabilities associated with different switching times

Consider a CDS contract that pays one monetary unit in case of default and zero otherwise. The decision of the firm to switch the two assets will have an impact on the value of the CDS contract, because it directly affects the generated cash available for reimbursing the debt, hence the default probability. Indeed, for a switching strategy θ, the the CDS contract pays at time T the amount:

$$C^\theta_T := \mathbf{1}_{A^\theta_T < N}.$$

In th is section we find expressions of the default probability $\mathbf{P}(A^\theta_T < N|\mathcal{F}_t)$ associated with a switching time θ of the assets of the firm. We will then use this result in order to characterize some incentives of the holder of a naked CDS contract. To begin, let us denote:

$$\begin{aligned}
\phi_1(t,x) &:= \mathbf{P}\left(\int_t^T e^{-ru} S^1_u du \leq x \middle| S^1_t = 0\right) \\
\phi_2(t,x) &:= \mathbf{P}\left(\int_t^T e^{-ru} S^2_u du \leq x \middle| S^2_t = 0\right).
\end{aligned}$$

Proposition 4.1. *The $\mathcal{F}_t$-conditional default probability associated with the switching time $\theta \geq t$ is given by:*

$$\mathbf{P}(A^\theta_T < N|\mathcal{F}_t) = \phi_1(t, \xi^1_t) + \underbrace{\mathbf{E}\left[\mathbf{1}_{\theta \leq T}\left(\phi_2(\theta, \xi^2_\theta) - \phi_1(\theta, \xi^1_\theta)\right)|\mathcal{F}_t\right]}_{\text{switching premium}}, \tag{5}$$

where:

$$\begin{aligned}
\xi^1_t &:= N - xh(0) - \int_0^t h(u) dS^1_u = N - \int_0^T S^1_{u\wedge t} e^{-ru} du \\
\xi^2_t &:= \xi^1_t + h(t) Z_t \\
Z_t &:= S^1_t - S^2_t \\
h(t) &:= \int_{t\wedge T}^T e^{-ru} du = \frac{e^{-r(t\wedge T)} - e^{-rT}}{r},
\end{aligned}$$

and $\phi_1(t,x)$ and $\phi_2(t,x)$ are respectively solutions of:

$$\frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_1}{\partial x}\mu_1 h(t) + \frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2}\sigma_1^2 h(t)^2 = 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R} \tag{6}$$
$$\phi_1(T,x) = 1_{x \geq 0} \quad \forall x \in \mathbb{R}$$
$$\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_2}{\partial x}\mu_2 h(t) + \frac{1}{2}\frac{\partial^2 \phi_2}{\partial x^2}\sigma_2^2 h(t)^2 = 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R} \tag{7}$$
$$\phi_2(T,x) = 1_{x \geq 0} \quad \forall x \in \mathbb{R}.$$

In particular, if the firm does not exchange the investment before time T, the default probability is:

$$\mathbf{P}(a_T^1 < N|\mathcal{F}_t) = \phi_1(t, \xi_t^1).$$

Proof. Let us denote:

$$\xi_t^3 := N - yh(0) - \int_0^t h(u) dS_u^2$$

and recall that

$$\phi_1(t,x) := \mathbf{P}\left(\int_t^T e^{-ru} S_u^1 du \leq x \big| S_t^1 = 0\right)$$
$$\phi_2(t,x) := \mathbf{P}\left(\int_t^T e^{-ru} S_u^2 du \leq x \big| S_t^2 = 0\right)$$

We have:

$$\begin{aligned}
&\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) = \mathbf{E}\left[\mathbf{1}_{\theta > T}\mathbf{1}_{a_T^1 \leq N} + \mathbf{1}_{\theta \leq T}\mathbf{1}_{a_\theta^1 + a_T^2 - a_\theta^2 \leq N}\big|\mathcal{F}_t\right] \\
&= \mathbf{E}\left[\mathbf{1}_{a_T^1 \leq N} + \mathbf{1}_{\theta \leq T}\left(\mathbf{1}_{a_\theta^1 + a_T^2 - a_\theta^2 \leq N} - \mathbf{1}_{a_T^1 \leq N}\right)|\mathcal{F}_t\right] \\
&= \mathbf{P}(a_T^1 \leq N|\mathcal{F}_t) + \mathbf{E}\left[\mathbf{1}_{\theta \leq T}\left\{\mathbf{P}\left(a_\theta^1 + a_T^2 - a_\theta^2 \leq N|\mathcal{F}_\theta\right) - \mathbf{P}\left(a_T^1 \leq N|\mathcal{F}_\theta\right)\right\}\Big|\mathcal{F}_t\right]
\end{aligned}$$

Furthermore:

$$\begin{aligned}
\mathbf{P}(a_T^1 \leq N|\mathcal{F}_t) &= \mathbf{P}\left(\int_t^T e^{-ru}(S_u^1 - S_t^1) du \leq N - \int_0^t e^{-ru} S_u^1 du - S_t^1 (e^{-rt} - e^{-rT})\frac{1}{r})|\mathcal{F}_t\right) \\
&= \mathbf{P}\left(\int_t^T e^{-ru}(S_u^1 - S_t^1) du \leq N + \int_0^t S_u^1 dh(u) - S_t^1 h(t)|\mathcal{F}_t\right) = \phi_1(t, \xi_t^1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{P}\left(a_t^1 + a_T^2 - a_t^2 \leq N|\mathcal{F}_t\right) &= \mathbf{P}\left(\int_t^T e^{-ru}(S_u^2 - S_t^2) du \leq N + \int_0^t S_u^1 dh(u) - S_t^2 h(t)|\mathcal{F}_t\right) \\
&= \mathbf{P}\left(\int_t^T e^{-ru}(S_u^2 - S_t^2) du \leq \xi_t^1 + (S_t^1 - S_t^2) h(t)|\mathcal{F}_t\right) \\
&= \phi_2(t, \xi_t^2)
\end{aligned}$$

Therefore,

$$\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) = \phi_1(t,\xi_t^1) + \mathbf{E}\left[\mathbf{1}_{\theta\le T}\left(\phi_2(\theta,\xi_\theta^2) - \phi_1(\theta,\xi_\theta^1)\right)|\mathcal{F}_t\right]$$

It remains to show that ϕ_1 and ϕ_2 satisfy respectively the equations (6) and (7) (the associated boundary conditions are easy to check).

Let us treat first the case of ϕ_1. Notice that the process:

$$\mathbf{P}(a_T^1 \le N|\mathcal{F}_t) = \phi_1(t,\xi_t^1)$$

is a martingale.

It is immediate from its definition that $\phi_1 \in C^{1,2}$ and is increasing in x. Therefore, one can apply Itô's lemma and impose the drift term of the process $\phi_1(t,\xi_t^1)$ to be null. Notice that ξ_t^1 satisfies the following SDE:

$$d\xi_t^1 = -\mu_1 h(t)dt - \sigma_1 h(t)dW_t^1,$$

and therefore:

$$d\phi_1(t,\xi_t^1) = \left(\frac{\partial\phi_1}{\partial t}(t,\xi_t^1) - \frac{\partial\phi_1}{\partial x}(t,\xi_t^1)\mu_1 h(t) + \frac{1}{2}\frac{\partial^2\phi_1}{\partial x^2}(t,\xi_t^1)\sigma_1^2 h(t)^2\right)dt - \frac{\partial\phi_1}{\partial x}(t,\xi_t^1)\xi_t^1\sigma_1 dW_t^1$$

The PDE (6) follows when imposing the drift above to be null.

Similarly, the process $\mathbf{P}(a_T^2 \le N|\mathcal{F}_t) = \mathbf{P}\left(\int_t^T e^{-ru}(S_u^2 - S_t^2)du \le \xi_t^3|\mathcal{F}_t\right) = \phi_2(t,\xi_t^3)$ is a martingale. The equation for ϕ_2 is then derived using the same arguments as in the case of ϕ_1. □

Below we give an explicit expression for the function ϕ^2. Both the PDE characterization above and the expression below will be useful later on.

Proposition 4.2. *The $\mathcal{F}_t$-conditional default probability associated with a switching time θ, $t \le \theta < T$, is given by:*

$$\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) = \mathbf{E}\left[\mathcal{N}\left(\frac{\xi_\theta^2 + \mu_2\int_\theta^T h(u)du}{\sigma_2\sqrt{\int_\theta^T h(u)^2du}}\right)\Big|\mathcal{F}_t\right], \tag{8}$$

where $\mathcal{N}(x)$ is the distribution function of a standard Gaussian random variable. It follows that:

$$\phi_2(t,x) = \mathcal{N}\left(\frac{x + \mu_2\int_t^T h(u)du}{\sigma_2\sqrt{\int_t^T h(u)^2du}}\right), \quad \textit{for } t \in [0,T).$$

Proof. Using integration by parts (notice that $h(T) = 0$), we have that:

$$-\int_t^T (S_u^2 - S_t^2)dh(u) = \int_t^T h(u)dS_u^2 = \mu_2\int_t^T h(u)du + \sigma_2\int_t^T h(u)dW_u^2.$$

The result then follows easily from Proposition 4.1, since for any stopping time θ, $\int_\theta^T h(u)dW_u^2$ is independent from $\mathcal{F}_\theta$ and Gaussian distributed, with zero mean and variance $\int_\theta^T h(u)^2du$.
□

Remark. The three dimensional process $(t, \xi_t^1, Z_t)_{t\geq 0}$ is a Markov process with state space $E := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Furthermore, (ξ_t^1) can (and will) be interpreted as an insolvability indicator at time t: $\xi_t^1 = N - \int_0^T S_{u\wedge t}^1 e^{-ru} du$ i.e., difference between the liability due time T and the cash account under the scenario that from t to T the cash flows are constant and equal to S_t^1. Thus ξ_t^1 positive can be interpreted as the firm being insolvable as seen from time t and the higher ξ_t^1 the more insolvable the firm is. Also, when Z_θ is positive the switching option is exercised with a loss.

Finally, let us notice that switching the investment at or after time T does not impact the time T default probability, that is: $\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) = \mathbf{P}(A_T^{\theta\wedge T} \leq N|\mathcal{F}_t)$.

Using the facts emphasized the previous remark in formula (5) leads to the following more simple expressions of the default probability:

Lemma 4.3. *The $\mathcal{F}_t$-conditional default probability for $t \leq T$, associated with the switching time $\theta \geq t$ is given by:*

$$\begin{aligned}\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) &= \mathbf{E}\left[\phi_2(\theta\wedge T, \xi_{\theta\wedge T}^2)|\mathcal{F}_t\right]\\ &= \mathbf{E}\left[\phi_2(\theta\wedge T, \xi_{\theta\wedge T}^1 + h(\theta\wedge T)Z_{\theta\wedge T})|(t, \xi_t^1, Z_t)\right].\end{aligned}$$

From the perspective of a naked CDS holder, the maximal value for the CDS contracts at time t can be achieved if a switching time is chosen by the firm such that it is a solution for:

$$ess \sup_{\theta\in\mathcal{T}_t} \mathbf{P}(A_T^\theta < N|\mathcal{F}_t).$$

The proposition below establishes a partial comparison for the default probabilities associated with different switching times. We can deduce that, in order to achieve the maximum default probability, it is always optimal to exercise the switching option with a loss (, i.e., $Z_t > 0$ in the stopping set). This is a consequence of the fact that waiting is always optimal in the profit zone (i.e., when $Z_t < 0$):

Proposition 4.4. *Suppose we have two stopping times η and θ such that $t \leq \eta < \theta \leq T_0(t)\wedge T$. Then, the following hold:*

(1) *If $Z_t = S_t^1 - S_t^2 > 0$, then:*

$$\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) < \mathbf{P}(A_T^\eta < N|\mathcal{F}_t).$$

(the sooner we switch the investment, the higher the probability of default; this works only on $[t, T_0(t)\wedge T]$).

(2) *If $Z_t = S_t^1 - S_t^2 < 0$, then:*

$$\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) > \mathbf{P}(A_T^\eta < N|\mathcal{F}_t).$$

(instead, on $[t, T_0(t)\wedge T]$, the later we switch the investment, the higher the probability of default).

Proof. We use the same notation as in the proof of Proposition 4.1.

The results are a consequence of the fact that on $[t, T_0(t)]$, the process $\phi_2(t, \xi_t^2), t \geq 0$ is either a strict supermartingale (if $Z_t > 0$) or a strict submartingale (if $Z_t < 0$). Indeed, since $\xi_t^2 = \xi_t^3 - \int_0^t e^{-ru} Z_u du$, we have:

$$d\xi_t^2 = \left(-e^{-rt} Z_t - \mu_2 h(t)\right) dt - \sigma_2 h(t) dW_t^2,$$

hence, (using first Itô's lemma, then equation (7)):

$$\begin{aligned} d\phi_2(t, \xi_t^2) &= \frac{\partial \phi_2}{\partial t}(t, \xi_t^2) + \frac{\partial \phi_2}{\partial x}(t, \xi_t^2) \left\{ \left(-e^{-rt} Z_t - \mu_2 h(t)\right) dt - \sigma_2 h(t) dW_t^1 \right\} + \frac{\partial^2 \phi_2}{\partial x^2}(t, \xi_t^2) h(t)^2 \sigma_2^2 dt \\ &= -\frac{\partial \phi_2}{\partial x}(t, \xi_t^2) e^{-rt} Z_t dt - \frac{\partial \phi_2}{\partial x}(t, \xi_t^2) h(t) \sigma_2 dW_t^2. \end{aligned}$$

To conclude, notice that $\frac{\partial \phi_2}{\partial x} \geq 0$ and furthermore if $Z_t < 0$ then $Z_s < 0$ for $s \in [t, T_0(t))$, respectively, if $Z_t > 0$ then $Z_s > 0$ for $s \in [t, T_0(t))$. $\square$

5. Optimal value of the investor's portfolio when holding a CDS position

We suppose that at time 0 the investor can choose to invest part of the available cash K in CDS, thus making a bet on its own investment into the firm's debt to default. We assume that the CDS position of the investor is not observed by market participants instantaneously time 0, when the transaction occurs (for instance off balance sheet investments). This is a reasonable assumption, in the cases where the CDS are traded over the counter and the trades are rather opaque (for a description of the functioning of the CDS markets, see Stulz [21]). The amount K should be interpreted as the maximum amount one can invest in buying CDSs at time 0 without having a market impact.

It follows that the anticipated switching time of the investments is T_ℓ. Hence, the market price at time 0 of a CDS contract that pays one monetary unit at the default event is given by (using Proposition 4.1):

$$C_0 := e^{-rT} \mathbf{P}(A_T^{T_\ell} < N) = e^{-rT} \mathbf{E}(\phi_2(T_\ell \wedge T, \xi_{T_\ell \wedge T}^2)),$$

where T_ℓ is the switching time that maximizes the firm's value.

If a proportion π of the available cash K is invested in CDS, it means the investor buys

$$n(\pi) := \frac{\pi K}{C_0}$$

CDS contracts and $(1 - \pi)K$ is kept as cash.

As already mentioned Section 2, the default of the firm occurs if at the maturity time of the debt T, the available cash A_T^θ generated by the investment is insufficient to reimburse the nominal value of the debt N. This is in general different from the default event $\{A_T^{T_\ell} < N\}$, which is anticipated by the market participants at time 0. Therefore at time T, the true payoff of the CDS is $\mathbf{1}_{\{A_T^\theta < N\}}$, i.e., it depends on the chosen switching strategy θ.

The value of the investor's portfolio with the strategies (π, θ), where $\theta \in \mathcal{T}_0$ is a switching time from the investment 1 to the investment 2 and $\pi \in [0, 1]$ the proportion of the available cash invested in CDS at time 0 is given by:

$$v_t^{\pi,\theta} = \alpha V_t^\theta + (1-\pi)Ke^{rt} + n(\pi)\mathbf{P}(A_T^\theta < N|\mathcal{F}_t).$$

We recall that V_t^θ is the firm's value corresponding to the switching strategy θ (see equation (1)) and the constant α is the proportion of the firm own by the investor. Note that the CDS are bought time 0 and then hold until the maturity T, while the switching decision can be made at any date $\theta \in \mathcal{T}_0$.

5.1. **The optimization problem with possibilities of CDS investment.** We search for the optimal strategies $(\theta, \pi) \in \mathcal{T}_0 \times [0,1]$, which solve the problem:

$$v^* = \sup_{(\theta,\pi)} e^{-rT}\mathbf{E}[v_T^{\pi,\theta}] = \sup_{(\theta,\pi)} e^{-rT}\mathbf{E}[\alpha V_T^\theta + (1-\pi)Ke^{rT} + n(\pi)\mathbf{1}_{\{A_T^\theta < N\}}]. \tag{9}$$

Our aim is to determine whether there are such strategies $(\theta, \pi) \in \mathcal{T}_0 \times [0,1]$ that can lead to a higher portfolio value for the investor than without the trading in CDS, or, stated otherwise, if the following holds:

$$\sup_{(\theta,\pi)} e^{-rT}\mathbf{E}[v_T^{\pi,\theta}] > \alpha V_0 + K = \sup_{\theta} e^{-rT}\mathbf{E}[v_T^{0,\theta}].$$

In other words, we shall compare bidimensional strategies (consisting in investing in CDS and switching the firm's assets) with unidimensional strategies (without the possibility of investing in CDS).

Below, we analyze the problem (9) and show that after some transformations, it is in fact equivalent to a classical optimal stopping problem, hence can be written in the form: $\sup_{\theta\in\mathcal{T}_0} \mathbf{E}\left[G_\theta\right]$ for some gain or reward process (G_t). The optimal level of the CDS investment is always $\pi = 1$, that is, the investor will optimally invest all the available cash in CDS. But, importantly, indifference zones exist, where the CDS position does not produce any additional gain in average. In this situations, $\pi = 0$ is also optimal.

Let us first have a look to the following sub-problem:

$$J(\theta,\pi) := \mathbf{E}[e^{-rT}n(\pi)\mathbf{1}_{\{A_T^\theta < N\}} - \pi K]$$

and

$$j(\theta) := \sup_{\pi\in[0,1]} J(\theta,\pi). \tag{10}$$

Then, one can notice that our original problem (9) is equivalent to:

$$\sup_{(\theta,\pi)\in\mathcal{T}_0\times[0,1]} e^{-rT}\mathbf{E}[v_T^{\pi,\theta}] = K + \sup_{\theta\in\mathcal{T}_0} \mathbf{E}[e^{-rT}\alpha V_T^\theta + j(\theta)] \tag{11}$$

Lemma 5.1. *Suppose that a switching time θ is fixed. The solution of the problem (10) is:*

$$j(\theta) = \max\{J(\theta,0), J(\theta,1)\} = J(\theta,1)^+$$

Therefore, at time 0 it is either optimal to invest all the cash or nothing in CDS. More precisely, if:

(1) $\mathbf{P}(A_T^\theta < N) > \mathbf{P}(A_T^{T_\ell} < N)$ *then,* $j(\theta) = J(\theta, 1)$*, hence it is optimal to invest all the cash in CDS (the market price of the CDS is undervalued);*
(2) $\mathbf{P}(A_T^\theta < N) < \mathbf{P}(A_T^{T_\ell} < N)$ *then,* $j(\theta) = J(\theta, 0)$*, hence it is optimal to hold entirely the cash and make no investment in CDS;*
(3) $\mathbf{P}(A_T^\theta < N) = \mathbf{P}(A_T^{T_\ell} < N)$ *then,* $j(\theta) = J(\theta, 1) = J(\theta, \pi) = J(\theta, 0)$*, any proportion* π *is optimal.*

Proof. Let us rewrite the function J as:

$$\begin{aligned} J(\theta, \pi) &= \frac{\pi K e^{-rT}}{C_0} \mathbf{P}(A_T^\theta < N) - \pi K \\ &= \pi K \left(\frac{\mathbf{P}(A_T^\theta < N)}{\mathbf{P}(A_T^{T_\ell} < N)} - 1 \right). \end{aligned}$$

Then:

$$J(\theta, \pi) \le K \left(\frac{\mathbf{P}(A_T^\theta < N)}{\mathbf{P}(A_T^{T_\ell} < N)} - 1 \right) = J(\theta, 1) \text{ when } \mathbf{P}(A_T^\theta < N) > \mathbf{P}(A_T^{T_\ell} < N);$$

respectively:

$$J(\theta, \pi) \le 0 = J(\theta, 0) \text{ when } \mathbf{P}(A_T^\theta < N) > \mathbf{P}(A_T^{T_\ell} < N).$$

□

As we shall see further on, the stopping times that lead to the inequality in point 2) of Lemma 5.1 (i.e., such that $\mathbf{P}(A_T^\theta < N) < \mathbf{P}(A_T^{T_\ell} < N)$) are never optimal in (9). All the other stopping times, i.e., the ones leading to the inequalities in 1) resp. 3) can be optimal.

5.2. **An alternative expression for the optimization problem (9).** For any fixed stopping time θ, let us introduce the following (uniformly integrable) martingale:

$$\begin{aligned} M_t^\theta : &= K + \alpha \mathbf{E}\left[\int_0^\theta e^{-ru} S_u^1 du + \int_\theta^\infty e^{-ru} S_u^2 du | \mathcal{F}_t \right] \\ &= K + e^{-rt} \alpha A_t^\theta + \mathbf{1}_{\theta > t} \alpha \left(e^{-rt} v_t^1 + \mathbf{E}[e^{-r\theta}(v_\theta^2 - v_\theta^1) | \mathcal{F}_t] \right) + \mathbf{1}_{\theta \le t} e^{-rt} \alpha v_t^2 \\ &= K + e^{-rt} \alpha V_t^\theta. \end{aligned}$$

In particular, for $\theta = T_\ell$ and $t = 0$ we get:

$$M_0^{T_\ell} = K + \alpha \left(v_0^1 + \mathbf{E}[e^{-rT_\ell}(v_{T_\ell}^2 - v_{T_\ell}^1)] \right) = \sup_{\theta \in \mathcal{T}_0} M_0^\theta.$$

Also, we can notice that:

$$v_t^{\pi,\theta} = M_t^\theta e^{rt} - \pi K e^{rt} + n(\pi) \mathbf{P}(A_T^\theta < N | \mathcal{F}_t)$$

Therefore, using the martingale property of M^θ as well as Lemma 5.1, the problem (9) can be written as:

$$
\begin{aligned}
v^* = \sup_{(\theta,\pi)\in\mathcal{T}_0\times[0,1]} e^{-rT}\mathbf{E}[v_T^{\pi,\theta}] &= \sup_{(\theta,\pi)\in\mathcal{T}_0\times[0,1]} e^{-rT}\mathbf{E}[M_T^\theta e^{rT} - \pi K e^{rT} + n(\pi)\mathbf{1}_{\{A_T^\theta<N\}}] \\
&= (\alpha v_0^1 + K) + \sup_{(\theta,\pi)\in\mathcal{T}_0\times[0,1]} \Big(\alpha\mathbf{E}[e^{-r\theta}(v_\theta^2 - v_\theta^1)] + J(\theta,\pi)\Big) \\
&= (\alpha v_0^1 + K) + \sup_{(\theta,\pi)\in\mathcal{T}_0\times\{0,1\}} \Big(\alpha\mathbf{E}[e^{-r\theta}(v_\theta^2 - v_\theta^1)] + J(\theta,\pi)\Big) \\
&= (\alpha v_0^1 + K) + \sup_{\theta\in\mathcal{T}_0} \alpha\mathbf{E}[e^{-r\theta}(v_\theta^2 - v_\theta^1)] + J(\theta,1)^+ \\
&= (\alpha v_0^1 + K) + \sup_{\theta\in\mathcal{T}_0} \alpha\mathbf{E}[e^{-r\theta}(v_\theta^2 - v_\theta^1)] + K\left(\frac{\mathbf{P}(A_T^\theta<N)}{\mathbf{P}(A_T^{T_\ell}<N)} - 1\right)^+
\end{aligned}
$$

To summarize, we need to solve the following infinite horizon problem:

$$
g_0 := \sup_{\theta\in\mathcal{T}_0} \underbrace{\alpha\mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)\right]}_{\text{gain/loss from switching the assets}} + \underbrace{\Big(n_0 e^{-rT}\mathbf{P}(A_T^\theta<N) - K\Big)^+}_{\text{gain from CDS}}, \tag{12}
$$

with $n_0 := \frac{K}{e^{-rT}\mathbf{P}(A_T^{T_\ell}<N)}$ representing the number of CDS that can be bought with the initial cash amount K.

5.3. **Transformation into a classical optimal stopping problem.** The problem (12) is not a classical optimal stopping problem, since it is not of the form: $\sup_{\theta\in\mathcal{T}_0}\mathbf{E}\left[G_\theta\right]$ for some gain or reward process (G_t). The next theorem shows that the problem can be transformed into a classical optimal stopping problem with finite horizon T.

Theorem 5.2. *The optimal portfolio value for the investor is:*

$$
v^* = K + \alpha v_0^1 + g_0,
$$

with:

$$
g_0 = \sup_{\theta\in\mathcal{T}_{0,T}} \mathbf{E}\left[G(\theta,\xi_\theta^1,Z_\theta)\right], \tag{13}
$$

with the gain function:

$$
\begin{aligned}
G(t,a,z) &= \alpha e^{-rt}\left(-\frac{z}{r} - \frac{(\mu_1-\mu_2)}{r^2}\right) + K\left(\frac{\phi_2(t,a+h(t)z)}{e^{rT}C_0} - 1\right) \qquad t\in[0,T) \\
G(T,a,z) &= \alpha e^{-rT} f\left(-\frac{z}{r} - \frac{(\mu_1-\mu_2)}{r^2}\right) + K\left(\frac{\phi_2(T,a+h(t)z)}{e^{rT}C_0} - 1\right),
\end{aligned}
$$

where (see the formula in Lemma 3.1):

$$
f(u) = Le^{a_1(u-L)},
$$

where $a_1 = (-m + \sqrt{m^2 + 2\sigma^2 r})/\sigma^2$, $L = \max(1/a_1, u)$ *and*

$$\phi_2(t,x) = \mathcal{N}\left(\frac{x + \mu_2 \int_t^T h(u)du}{\sigma_2 \sqrt{\int_t^T h(u)^2 du}}\right), \quad \textit{for } t \in [0,T).$$

Proof. See the appendix. □

6. Optimal investment behavior: numerical implementation

We implement the optimization problem (13) using the Longstaff-Schwartz algorithm (see the original paper by Longstaff and Schwartz [16], or Clément et al. [6]).

Our base case parameters are:

$$x = 100; \quad y = 95; \quad \mu_1 = 2; \quad \mu_2 = 1; \quad r = 0.05;$$
$$\sigma_1 = 7; \quad \sigma_2 = 8; \quad \rho = 0; \quad T = 1; \quad N = 100; \quad K = 200.$$

For simplicity we assume $\alpha = 1$. Notice that the optimal strategies for a general $\alpha \neq 0$ and cash $K\alpha$ are identical to the ones we study here since the objective function is this case is the one we study below (i.e., with $\alpha = 1$), times the constant α.

The optimal firm's value without CDS investment, corresponding to the base case parameters is:

$$V_0 := \frac{x + \mu_1/r}{r} + E\left[e^{-rT_\ell}(v^2_{T_\ell} - v^1_{T_\ell})\right]) = 2868.054.$$

The price of one CDS contract with maturity $T = 1$ year is 0.1853. The cash available for investment in CDS (K=200) is approximatively 7% the NPV of the first investment ($v^1_0 = 2800$) and allows to buy 1079 CDS contracts time 0, which is more that 10 times the number of contracts that will correspond to a complete insurance against default (i.e., 100 contracts).

In Figures 2–10 below, we plot the values functions and the default probabilities corresponding to the optimal strategies, when respectively the parameters K, N, μ_1, μ_2, σ_1, σ_2, ρ, r, x vary.

More precisely, on the left-hand side of each figure we plot:

(1) the value of the option to switch the firm's assets (lower, dotted curve):

$$f(v_0) = \sup_{\theta \in \mathcal{T}_0} \mathbf{E}\left[e^{-r\theta}(v^2_\theta - v^1_\theta)\right],$$

(2) the optimized value with CDS (middle, solid curve):

$$g_0 := \sup_{\theta \in \mathcal{T}_0} \mathbf{E}\left[e^{-r\theta}(v^2_\theta - v^1_\theta)\right] + \left(n_0 e^{-rT} \mathbf{P}(A^\theta_T < N) - K\right)^+, \tag{14}$$

(3) the CDS position (i.e., the number of CDS contracts that are optimally bought at time 0), which also indicate the gain in case of default, since each CDS contract

pays 1 dollar in case of default (upper, dashed curve):

$$n_0 = \frac{K}{C_0} = \frac{K}{e^{-rT}\mathbf{P}(A_T^{T_\ell} < N)}.$$

On the right-hand side of each figure, we plot the default probabilities:

(1) the anticipated default probability (lower, dotted curve):

$$p^e := \mathbf{P}(A_T^{T_\ell} < N) = \mathbf{E}\left[\phi_2(T_\ell, \xi_{T_\ell}^2)\right],$$

which is the default probability anticipated by the market investors;

(2) the optimal default probability p^* (middle, solid curve), which is the default probability that corresponds to the optimal switching time in (14).

(3) the maximal default probability that can be obtained by switching the firm's assets (upper, dashed curve). This is obtained as solution of:

$$\bar{p} := \sup_{\theta \in \mathcal{T}_{0,T}} \mathbf{P}(A_T^\theta < N) = \sup_{\theta \in \mathcal{T}_{0,T}} \mathbf{E}\left[\phi_2(\theta, \xi_\theta^2)\right];$$

The following relation holds:

$$p^e \le p^* \le \bar{p}.$$

The levels of the parameters from where the CDS position becomes profitable (in the sense that g_0 is greater that $f(v_0)$) are not always easy to identify visually in the pictures, since sometimes very small profits from CDS are obtained, and by exercising the switching option with a loss. In order to make them easy to identify, these levels are marked in each figure by vertical lines. Thus the pictures are split in two parts, on the one side we have strategies where the CDS position produces gains and on the other side we have no gains from the CDS position. The exception is Figure 6, where we did not identify levels of the parameter μ_1 where the CDS position has no gains (the other parameters being fixed).

Before commenting on each of the figures below, let us make some general remarks:

(i) in our pictures, the CDS position n_0 is never null (as long as $K > 0$) since we have shown in Section 5 that it is always optimal to invest all the available cash K in CDS. However, it exists an indifference zone, i.e., values of the parameters where the investor is in fact indifferent between holding the CDS position or cash because holding the CDS position does not allow the investor to obtain additional gains. In such situations $n_0 = 0$ is also optimal. In the figures below, these indifference zones can be identified as the areas when the curves $f(v_0)$ and g_0 coincide (as was explained in the paragraph above), or alternatively, the areas where the curves p^e and p^* coincide.

(ii) It is known from the previous section that in the indifference zone T_ℓ is optimal, meaning that no other switching time will make the profit higher. But T_ℓ might not be the only optimal stopping time and our numerical procedure does not allow us to characterize the optimal switching boundary. We make the choice to suppose that in the indifference zone the investor will implement the same strategies as without the CDS, meaning exercising at T_ℓ, since this is optimal (maybe among other optimal switching times). We also believe that in practice investors would not deviate from T_ℓ if there is no extra gains from doing so and might decide not to hold CDS at all, since in this indifference zones $\pi = 0$ is also optimal.

(iii) The main quantities determining the optimal strategies are:
  - The value of the option to switch the assets $f(v_0)$. This appears as the lower curve on the left-hand side of each figure. When the value of the switching option is sufficiently high, the investor will simply adopt the optimal strategy as without CDS. On the opposite, low values of the switching option incite the investor to switch the assets with a loss.
  - The anticipated default probability p^e: it determines the price of a CDS ($C_0 = p^e e^{-rT}$) and therefore how many CDS are initially bought. As mentioned already p^e appears as the lower curve on the right-hand side of each figure. Also, the corresponding size of the CDS position, appears in the left-hand side of each figure.

(iv) Most often the two above quantities ($f(v_0)$ and p^e) move together in the same direction. When the value of the switching option $f(v_0)$ is low, the first asset is advantageous (as compared to the second one), and the associated cash flows high, leading to a relatively low expected default probability p^e. An exception to this rule appears when the parameter μ_2 varies. In this case, high return on the second asset increases the value of the exchange option but decreases the default probability (since in this case, we get closer of the exchange barrier and the cash flows from the second asset contribute to reduce the default probability).

(v) Outside the indifference zones mentioned in (i) above, investors have aggressive strategies (i.e., that lead fast to high default probabilities, close to the maximal one).

(vi) A portfolio of CDS that corresponds to the levels of default insurance (that is, the number of CDS bought does not exceed N which is the level of the liability due time T) leads to optimal strategies that are the ones as without CDS.

(vii) More CDS in the investor's portfolio, implies higher deviations from the optimal strategies, in the sense that $p^* - p^e$ increases.

6.1. **Impact of the cash level K (market value of the CDS position) on the optimal strategies.** The impact of the cash level K on the optimal strategies for the base-case parameters is depicted in Figure 2. n_0 is linear in K. We observe that when K is 130 or less, there is no profit from the CDS position, that is, $f(v_0) = g_0$ (figure on the left hand side). Indeed, when K low, few CDS contracts can be bought and the investor will not have sufficient incentive to deviate from the optimal switching strategy at T_ℓ. This also means that the CDS are fairly priced (market expectations correspond to the investor's optimal strategy). However when K increases, the CDS position increases (green curve) and investors are incited to increase the default risk of the firm, buy switching the assets with a negative profit. In these cases the CDSs produce a high return: for instance when $K = 200$ (base case) the expected gain from CDS $g_0 - f(v_0)$ is 141.27. We see that the optimal default probability of the firm p^* increases steeply from an initial value of 19.48%

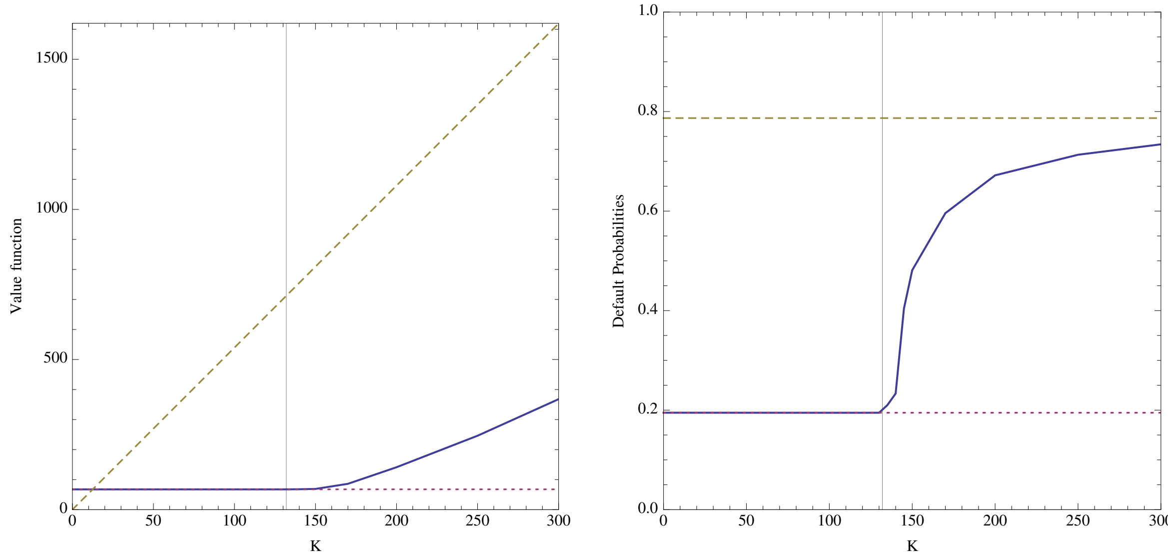

FIGURE 2. Impact of the cash level K on the optimal strategies: firm's value (left) and optimal default probability (right).

(without CDS) to a level of 73.5% which is close to the maximal default probability that could be implemented, which equals 78.7%.

6.2. **Impact of the debt level N on the optimal strategies.** The parameter N represents the value of the firm's liabilities that are due to be paid at time $T = 1$ (nominal value of the debt). It also represents the default barrier. As seen in Figure (3), left, the optimized gain with CDS g_0 decreases when N increases to finally reach the level of the switching option (which does not depend on the level of N and represents the optimized gain without CDS). Both the maximal default probability and the expected default probability increase when N increases. The optimal default probability is not monotone: when N low, CDS price is low hence the investor can buy many CDS and is tempted to implement a default probability close to the maximal one. As the level of N increases, the price of the CDS increases, less CDS can be bought at time 0 and the investor will find optimal to decrease the default probability up to the level of the expected one (this occurs near $N = 102$, when the number of CDS in the portfolio is 598). The most profitable for the investor is when the firm has low debt, hence low default probability: this permits to buy many CDS (al low price) and subsequently increase the default risk of the firm and the value of the CDS by switching the firm's assets with a loss.

6.3. **Impact of the assets volatilities σ_1 and σ_2 on the optimal strategies.** When the volatility σ_1 or σ_2 increases, the switching option has more value. The expected default probability p^e increases when σ_1 increases, so that the CDSs are becoming more expensive. In this situation, with high initial volatility (σ_1 more than 10) the CDS position generates

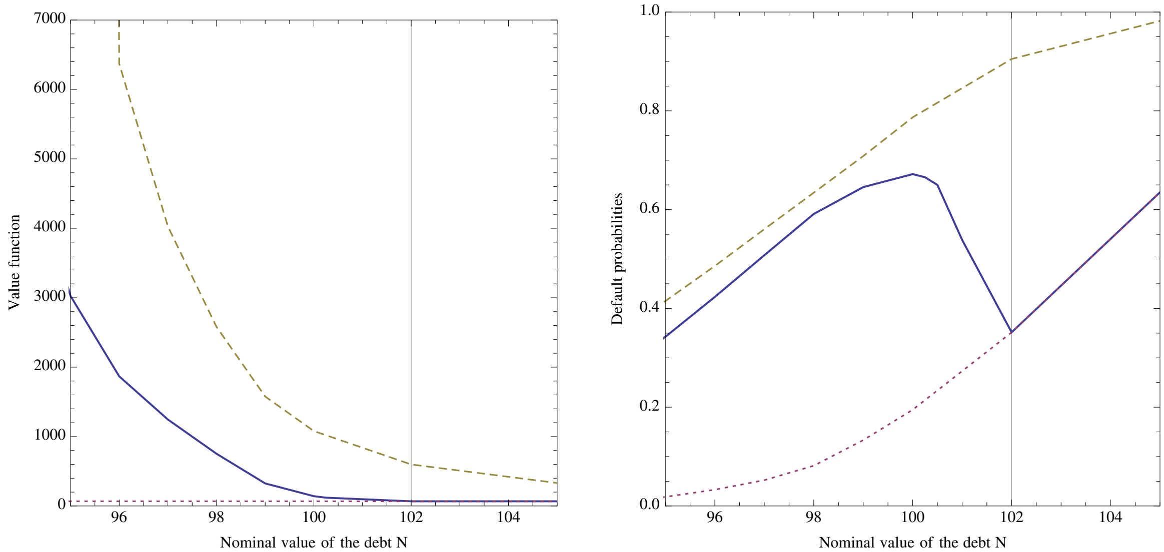

FIGURE 3. Impact of the debt level N on the optimal strategies: firm's value (left) and optimal default probability (right).

no expected gain ($f(v_0) = g_0$) and the optimal strategy is to switch the investments as without CDS, at T_ℓ, hence without increasing the default risk as compared to the expected level. In order to achieve high gains from the CDS position, the investor prefers a low initial volatility for the firm's cash flows. For instance, when σ_1 is 6, the optimized gain with CDS is 272.64 and the exchange option is 57.9. The default probabilities are $p^e = 15.6\%$, $p^* = 72.1\%$ and $\bar{p} = 77.78\%$.

Because at time 0 the underlying is rather far from the hitting level ℓ where the two assets are expected to be switched (more precisely, $\mathbf{P}(T_\ell \leq T)$ is very low for the parameters considered), the effect of σ_2 on the expected default probability is very low, and by consequence the price of the CDS is constant (the same thing holds when the expected return of the second asset μ_2 varies, see subsection below). Because the CDS position stays practically constant with higher σ_2, the indifference level (where $f(v_0) = g_0$) is reached with higher volatility compared with the case when σ_1 was considered (here we have σ_2 at least 20 for the CDS position to generate no expected gain)

6.4. **Impact of the assets expected returns μ_1 and μ_2 on the optimal strategies.** The impact of the expected return μ_1 is as follows: when μ_1 increases $f(v_0)$ decreases (less interesting to switch the first asset against the second); also the anticipated default probability decreases (more likely to accumulate sufficient cash for reimbursing the debt). For the parameters considered the optimal strategy with CDS is always different from the optimal strategy in absence of CDS. More precisely, we can observe the following:

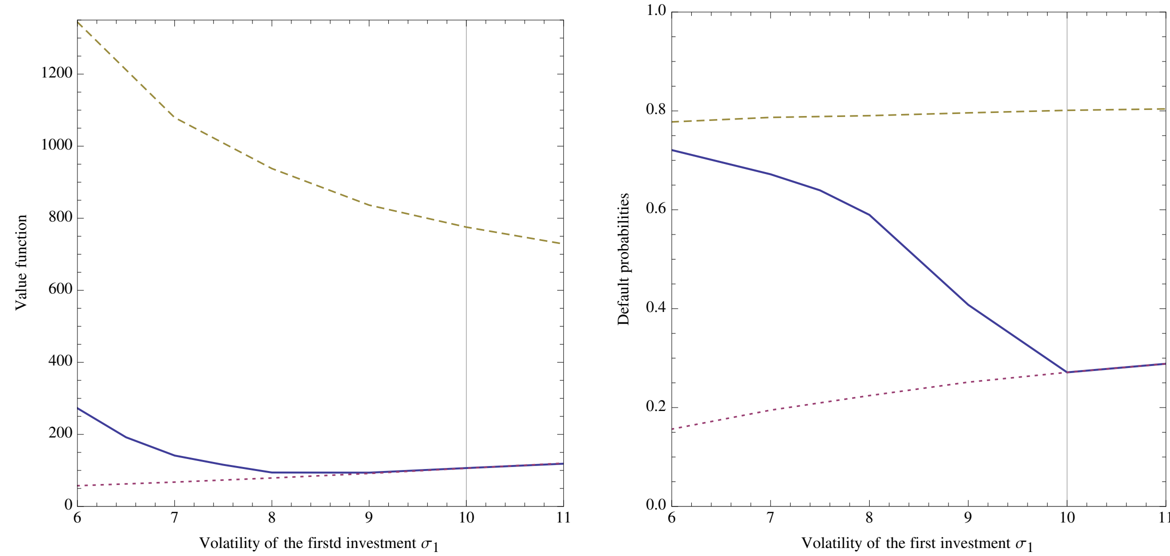

FIGURE 4. Impact of the volatility of the first investment σ_1 on the optimal strategies: firm's value (left) and optimal default probability (right).

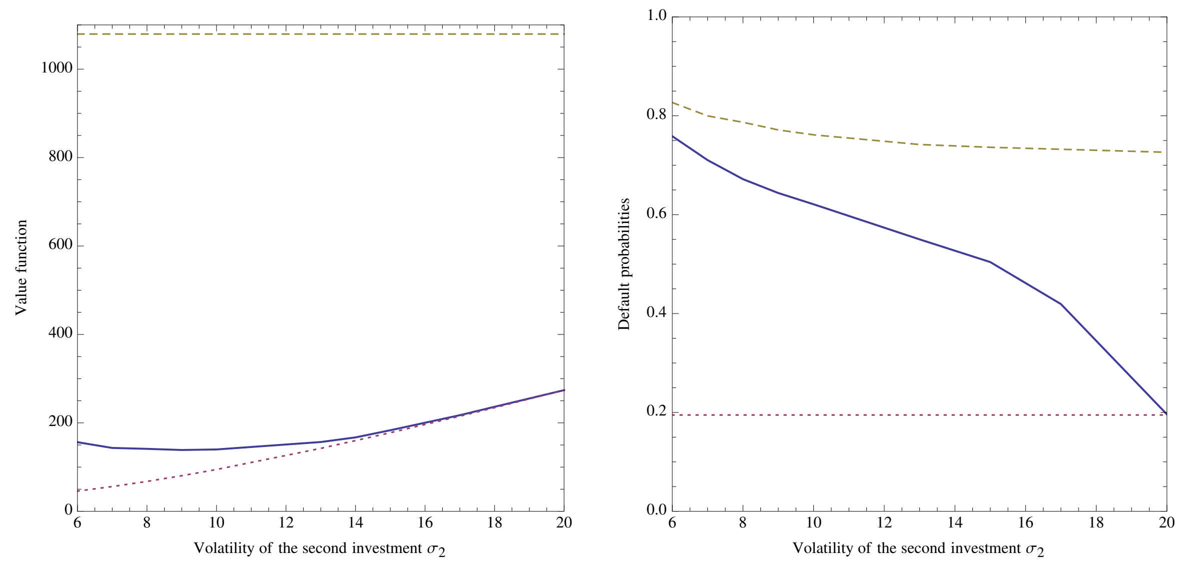

FIGURE 5. Impact of the volatility of the second investment σ_2 on the optimal strategies: firm's value (left) and optimal default probability (right).

- the scenario with low μ_1 corresponds to high value of the switching option, high expected default probability and hence low CDS position. The market anticipated switching time T^ℓ is closer: the optimal barrier ℓ is closer to reach by the process Z. The investor's optimal strategy consists mainly in not switching the assets once the barrier ℓ is hit, hence keeping the first asset longer. This strategy increases the probability of default as compared to the anticipated one;
- the scenario with high μ_1 corresponds both to low value of the switching option and high CDS position. This combination incites the investor to implement a high optimal default probability mostly by switching the assets with a loss.

The non-monotonicity of the value function and optimal default probability can be explained as follows. An increase in the parameter μ_1 has a double effect. First, less of sample paths of the process Z actually reach the barrier ℓ and more sample paths reach a high losses region. Secondly, the CDS contract is cheaper, hence the CDS position increases. Therefore, as μ_1 increases, even though the investor becomes more eager to increase the default probability, he actually becomes less able to implement the strategy of postponing the switching decision at $Z = \ell$. Instead, gradually, the strategy of premature exercising with high losses becomes more effective. It is not always the case that one effect dominates the other, hence the non-monotonicity of the above mentioned functions.

For instance, when $\mu_1 = 7$: (i) there are 3285 CDS in the portfolio, that is more than 32 times the level of a complete insurance, (ii) the switching option is almost null $f(v_0) \sim 0$, (iii) the expected default probability is 6.4%, (iv) the optimal default probability is 45%, (v) the expected profit from the CDS position is $g_0 - f(v_0) = 96.55$ which is approx. 48% of the initial investment in CDS ($K = 200$).

As a comparison, when $\mu_1 = 1$ (i.e, the same value as for μ_2): (i) there are 915 CDS in the portfolio, that is almost 9.2 times the level of a complete insurance, (ii) the value of the switching option is $f(v_0) = 212$, (iii) the optimal default probability is 73.9%, (iv) the expected profit from the CDS position: $g_0 - f(v_0) = 148$ which is 74 % of the initial investment in CDS.

Notice that $\mu_1 = 4.5$ is the parameter that leads to the lowest optimal default probability ($p^* = 26\%$) and closest to the one without CDS ($p^e = 12\%$).

As mentioned already, for the parameters considered the optimal strategy with CDS is never the same as without CDS. This is because the CDS position is always very important, for all ranges of μ_1 considered. But if we consider a lower CDS investment, for instance $K = 100$, we get that the optimal strategy with CDS is the same as without CDS as long as $\mu_1 \geq 2$ (other parameters being as in the base case).

Concerning the expected return of the second asset μ_2, the effects are: $f(v_0)$ and μ_2 move in the same direction, but very little effect on the expected default probability (and hence on the size of the CDS position), which decreases when μ_2 increases. We are in a situation different from the usual setting: here the indifference point, where there is no profit from the CDS position (the investor implements the optimal strategy as without the CDS) is attained for when the exchange option has low value (here this occurs for $\mu_2 \leq 0$).

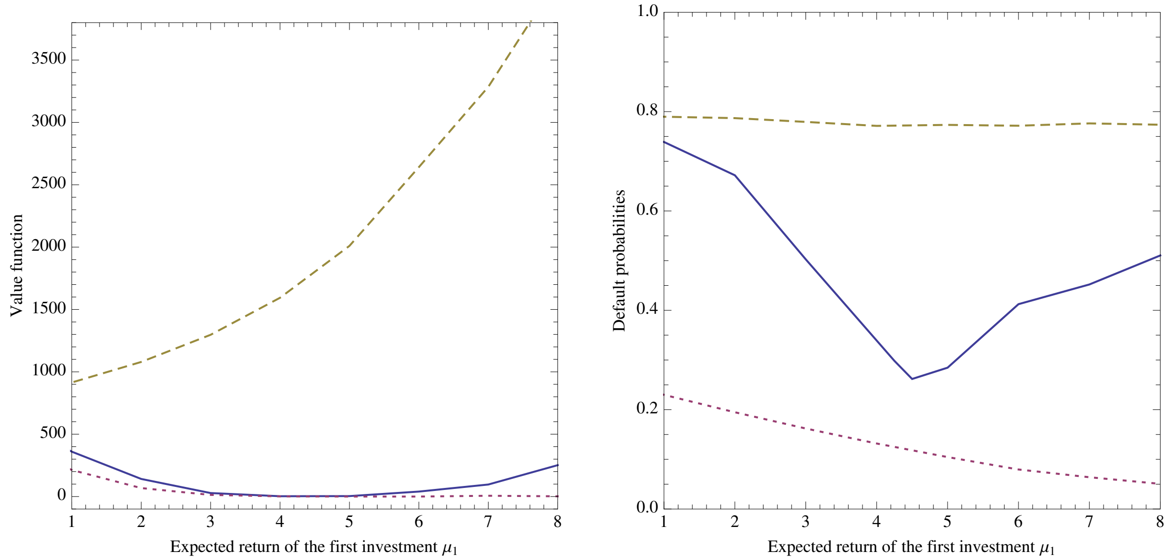

FIGURE 6. Impact of the level of the expected return of the first investment μ_1 on the optimal strategies: firm's value (left) and optimal default probability (right).

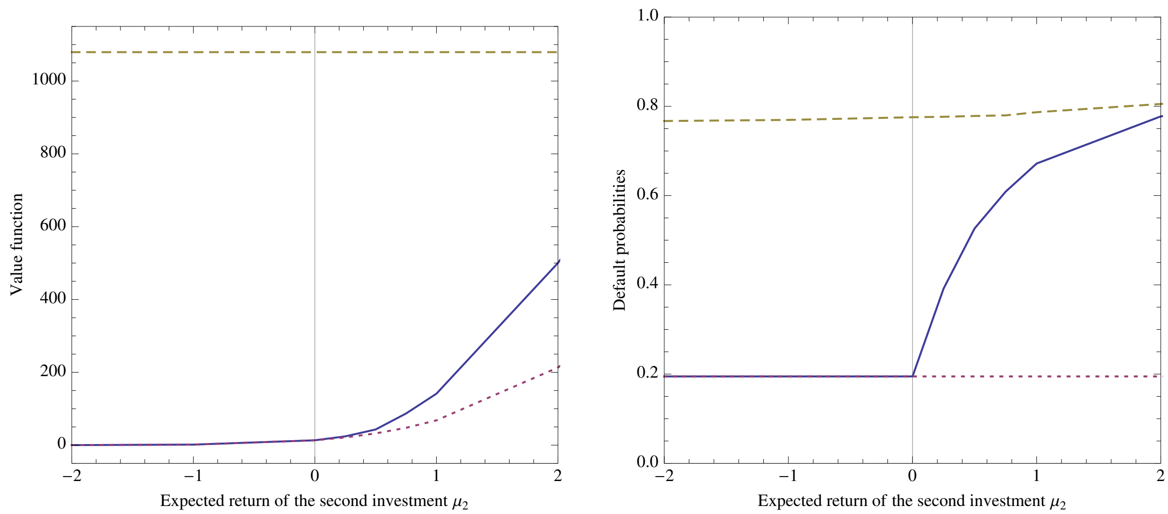

FIGURE 7. Impact of the level of the expected return of the second investment μ_2 on the optimal strategies: firm's value (left) and optimal default probability (right).

6.5. **Impact of the correlation coefficient ρ.** The correlation coefficient ρ between the two Brownian motions W^1 and W^2 determines the volatility of the process $Z = S^1 - S^2$: when $\rho = -1$ the volatility of Z is the highest, and equals $\sigma^1 + \sigma^2 = 15$, while for $\rho = 1$ the volatility is the lowest, $|\sigma^1 - \sigma^2| = 1$. As a consequence, the value of the exchange option decreases when ρ increases, as displayed in Figure 8. The profit function g_0 which includes the effect of the CDSs also decreases when ρ increases, however the decrease takes place at a slower rate, so that in fact the net profit from CDS given by $g_0 - f(v_0)$ increases. The value of the CDS position is $n_0 = 1079$ contracts, independent of the correlation (we did not display the curve for n_0, flat).

Concerning the default probabilities, the expected default probability does not much react to the change in correlation. This is due to our base case parameters. In fact, an increase of the correlation produces a double effect on the default probabilities. First, as already mentioned, the optimal switch of the assets occurs earlier, hence has more chances to occur before the maturity of the debt. The optimal switch brings in new cash at the switching time, hence decreases the default probability. The second effect has an inverse impact, as the second asset produces less cash on average than the first one ($\mu_2 < \mu_1$). For our base case parameters, these effects are not visible in Figure 8, because they have a small size hence seem perfectly compensating each other. But, if for instance we choose $\mu_1 = \mu_2 = 2$, the other parameters being the same, the expected default probability for $\rho = 1$ is 19.48% while for $\rho = -1$ it becomes 19.30%, hence a small effect becomes visible. With $\mu_1 = 2$ and $\mu_2 = 4$, the expected default probability decreases from 19.48% (for $\rho = 1$) to 18.25% (for $\rho = -1$).

The implemented default probability p^* is decreasing when the correlation coefficient increases, the investor being less able to produce losses to the firm.

6.6. **Impact of other variables: interest rate r and the initial value of the first asset x.** The impact of these variable is depicted in Figures 9 and 10. The behavior is the classical one: low r (resp. low x) leads to higher value of the exchange option and also higher expected default probability. In this case the optimal strategies with and without CDS are the same. As r (resp. x) increases, the exchange option loses value and the default probability is lower (hence CDS position higher). The investor implements optimal strategies that lead to higher default probabilities as r (reps. x) increases beyond the indifference point (this is when $r \geq 0.04$ resp. $x \geq 98$).

7. Conclusions

This paper analyses the possible drawbacks of large CDS positions in terms of investment strategies and risk taking incentives. Second best choices for investment decisions, could become first best in the presence of these products, when used mainly for speculative purposes.

In this paper, we attempt to apprehend these risks in a real options setting. Thus, we have quantified the default risk of a firm when a shareholder can at the same time influence its management decisions and hold CDSs. We have analyzed an optimal asset replacement problem for various sizes of the CDS position. In our paper, the investor is only allowed to

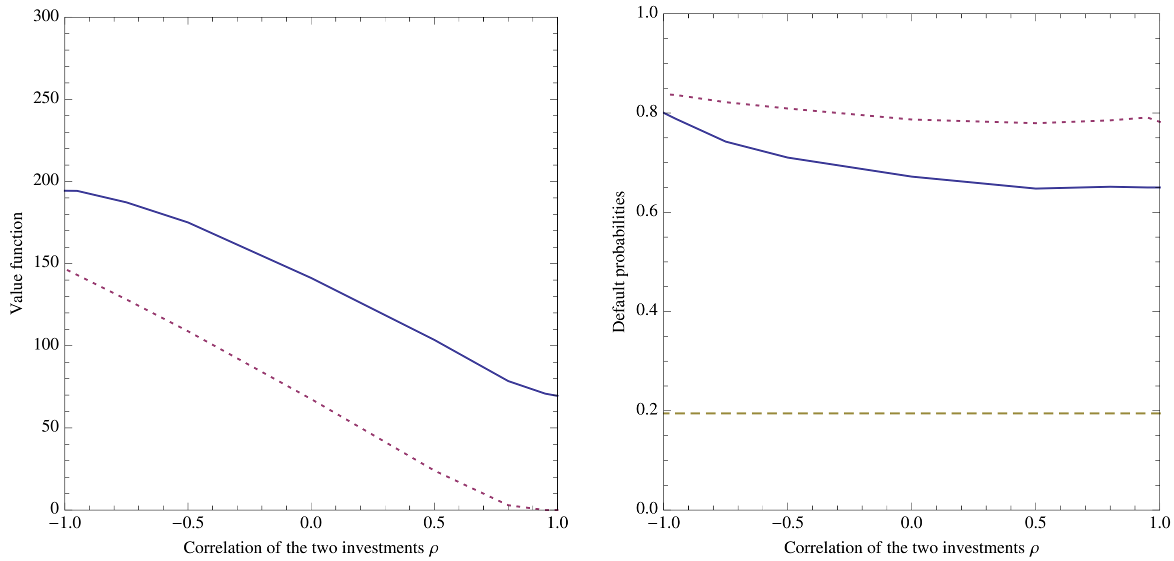

FIGURE 8. Impact of the level of the correlation coefficient ρ on the optimal strategies: firm's value (left) and optimal default probability (right).

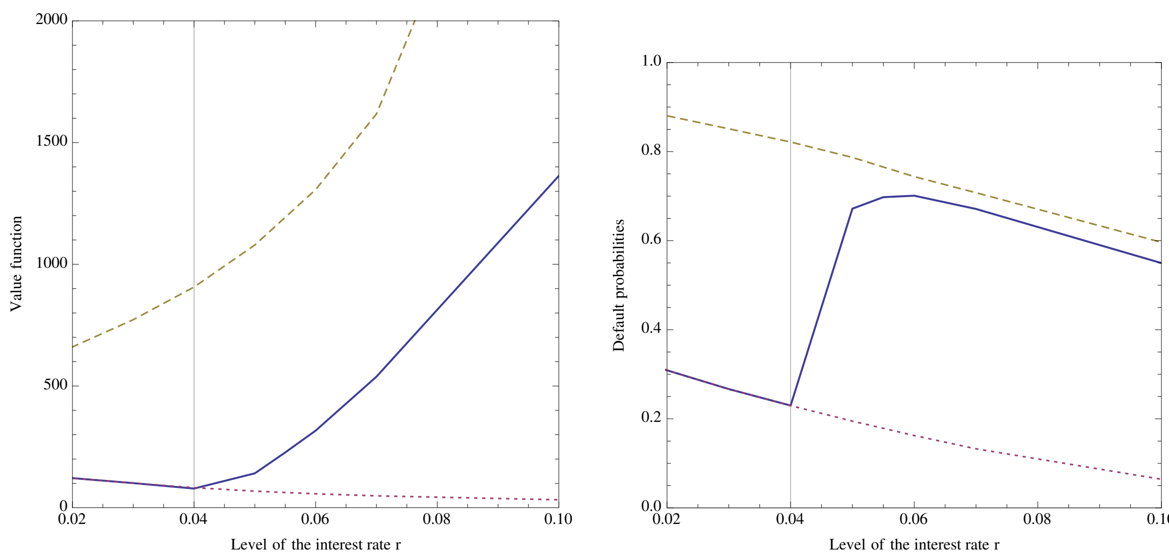

FIGURE 9. Impact of the level of the interest rate r on the optimal strategies: firm's value (left) and optimal default probability (right).

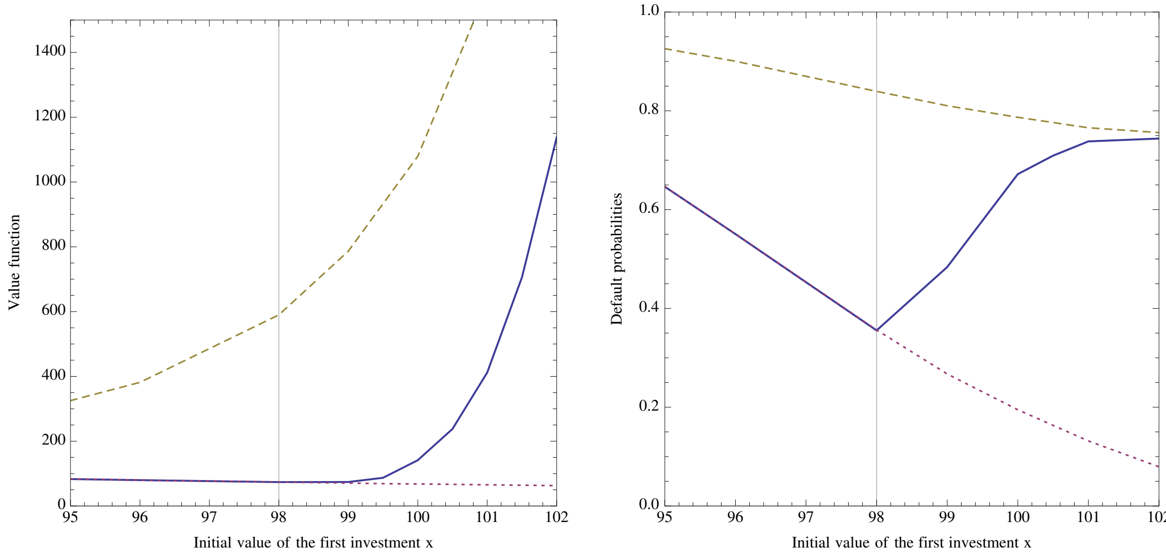

FIGURE 10. Impact of the initial value of the first investment x on the optimal strategies: firm's value (left) and optimal default probability (right).

invest some fixed amount of cash at most, since in practice, only a limited amount of CDSs can be bought at a given market price, that is, without having a market impact. Moreover, we assumed that the market price is an equilibrium price that corresponds to the maximal firm's value (i.e., when the management decisions are implemented in order to maximize the firm's value). Our main findings are that the default risk of the firms increases sharply in presence of large CDS portfolios, that exceed several times the levels of the complete insurance. However, when the size of the CDS portfolio is moderate we find that the CDS position does not alter the firm's default probability. There is no incentive in this case for the investor to invest in CDSs (he is in fact indifferent between investing or not) and if he invests, the default probability of the firm is not modified in presence of CDSs, within these indifference zones.

One interesting question for future research would be to analyze the optimal behavior for the investor when the market price for the CDSs is higher than the one we have considered. This can occur for instance when market participants anticipate some deviations from the optimal management strategy. Alternatively, one can argue that large positions of CDSs cannot be constructed at the cheapest price because of a limited supply at a given point in time. In this situation, we expect the investor to be sometimes discouraged from buying the CDSs (this would produce losses zones that would include and exceed our considered indifference zones). We do not investigate this situation in this paper, but we can expect that the investor would take a binary decision: either not invest in the CDSs at all (when he is not able to have a sufficiently large CDS position, hence is in the loss zone), or invest integrally his available cash (when this leads to an important CDS position) and implement much more aggressive strategies in terms of default risk.

Appendix

A. Proof of Lemma 3.1

The function f^{τ_c} is a solution of the following:

$$-rf + m\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = 0$$
$$\lim_{x \to c} f(x) = c$$
$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0$$

The general solution of the ODE above is given by: $f(x) = Ae^{a_1 x} + Be^{a_2 x}$, with a_1 and a_2 the roots of the equation $\frac{\sigma^2}{2}a^2 + ma = r$. One can check that indeed $a_{1,2} = (-m \pm \sqrt{m^2 + 2\sigma^2 r})/\sigma^2$, as stated in the lemma. Notice that $a_1 > 0$ and $a_2 < 0$. Using then the boundary conditions, one can find that indeed the function f^{τ_c} is as stated.

For maximizing over the possible levels of c, we need to distinguish the two cases $u \le c$ and $u \ge c$. Notice that for $c \le u$ we have $f^{\tau_c}(u) \le u = f^{\tau_u}(u)$, and hence:

$$f(u) = \max\{\sup_{c \ge u} f^{\tau_c}(u), \sup_{c \le u} f^{\tau_c}(u)\} = \max\{\sup_{c \ge u} ce^{a_1(u-c)}, u\}.$$

The supremum of the function $c \mapsto ce^{a_1(u-c)}$ is attained at $c = \frac{1}{a_1}$, hence the optimal switching time is: $L = \frac{1}{a_1}\mathbf{1}_{1/a_1 \ge u} + u\mathbf{1}_{1/a_1 \le u} = \max\{1/a_1, u\}$.

B. Proof of Theorem 5.2

The original problem (12) and the problem (13) in the theorem do not look the same: they do not have the same gain process, nor the same horizon. We are going to show that however, the two problems have the same sup, which is obtained using the same optimal exercising time.

We first show that the sup in (12) can be obtained by solving a classical optimal stopping problem with infinite horizon, then we apply the dynamic programming principle in order to transform it in a finite horizon problem.

We denote:

$$\tilde{G}_t := \alpha e^{-rt}\left(-\frac{Z_t}{r} - \frac{(\mu_1 - \mu_2)}{r}\right) + K\left(\frac{\phi_2(t \wedge T, \xi_t^2 \wedge T)}{\mathbf{P}(A_T^{T_\ell} < N)} - 1\right) \qquad t \in [0, \infty), \tag{15}$$

so that:

$$\mathbf{E}\left[\tilde{G}_\theta\right] = \alpha \mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)\right] + K\left(\frac{\mathbf{P}(A_T^\theta < N)}{\mathbf{P}(A_T^{T_\ell} < N)} - 1\right).$$

Notice that:

$$\begin{aligned}
g_0 &= \sup_{\theta\in\mathcal{T}_0} \alpha\mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)\right] + K\left(\frac{\mathbf{P}(A_T^\theta < N)}{\mathbf{P}(A_T^{T_\ell} < N)} - 1\right)^+ \\
&= \max\left\{\sup_{\theta\in\mathcal{T}_0} \alpha\mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)\right]; \sup_{\theta\in\mathcal{T}_0} \alpha\mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)\right] + K\left(\frac{\mathbf{P}(A_T^\theta < N)}{\mathbf{P}(A_T^{T_\ell} < N)} - 1\right)\right\} \\
&= \max\left\{\alpha\mathbf{E}\left[e^{-rT_\ell}(v_{T_\ell}^2 - v_{T_\ell}^1)\right]; \sup_{\theta\in\mathcal{T}_0} \mathbf{E}\left[\tilde{G}_\theta\right]\right\} \\
&= \max\left\{\mathbf{E}\left[\tilde{G}_{T_\ell}\right]; \sup_{\theta\in\mathcal{T}_0} \mathbf{E}\left[\tilde{G}_\theta\right]\right\} \\
&= \sup_{\theta\in\mathcal{T}_0} \mathbf{E}\left[\tilde{G}_\theta\right]
\end{aligned}$$

Therefore,

$$g_0 = \sup_{\theta\in\mathcal{T}_0} \mathbf{E}\left[\tilde{G}_\theta\right], \tag{16}$$

which is a classical optimal stopping problem with infinite horizon, and leads to the same optimal stopping time than the original formulation. Using the dynamic programming principle, we will now transform it in a finite horizon problem.

For this purpose, we generalize the problem (16) to all times $t \geq 0$:

$$\begin{aligned}
g_t :&= ess \sup_{\theta\in\mathcal{T}_{t,\infty}} \alpha\mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)|\mathcal{F}_t\right] + \left(n_0 e^{-rT}\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) - K\right) \\
&= ess \sup_{\theta\in\mathcal{T}_{t,\infty}} (f_t^\theta + h_t^\theta).
\end{aligned} \tag{17}$$

We have also introduced the notation:

$$f_t^\theta := \alpha\mathbf{E}\left[e^{-r\theta}(v_\theta^2 - v_\theta^1)|\mathcal{F}_t\right]$$

which is the gain/loss from switching the assets at the stopping time θ, and:

$$h_t^\theta := \left(n_0 e^{-rT}\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) - K\right).$$

which is the gain/loss from the CDS position.

Notice that for the situations where $T \leq t \leq \theta$ the gain form CDS stays constant and is independent of the switching time θ:

$$\begin{aligned}
h_t^\theta &= \left(n_0 e^{-rT}\mathbf{P}(A_T^\theta < N|\mathcal{F}_t) - K\right) = \left(n_0 e^{-rT}\mathbf{P}(A_T^T < N|\mathcal{F}_t) - K\right) \\
&= \left(n_0 e^{-rT}\mathbf{1}_{\{a_T^1 < N\}} - K\right) = h_T^\infty;
\end{aligned}$$

and we only have an option to switch the investment in the optimal stopping problem (17). Or, the solution of this problem has been already detailed in Section 3.

Hence from Section 3, for $t \geq T$, the optimal stopping time in the problem above is $T_\ell(T)$, in particular, for $t = T$ we have:

$$g_T = ess \sup_{\theta \in \mathcal{T}_{T,\infty}} (f_T^\theta + h_T^\infty) = f(v_T) + h_T^\infty = f_T^{T_\ell(T)} + h_T^\infty = G_T.$$

Hence, using the dynamic programming principle (see for instance Karatzas and Shreve [15], Appendix D), we obtain:

$$g_0 = \sup_{\theta \in \mathcal{T}_0} \mathbf{E}\left[\mathbf{1}_{\{\theta < T\}} \tilde{G}_\theta + \mathbf{1}_{\{\theta \geq T\}} g_T\right] = \sup_{\theta \in \mathcal{T}_{0,T}} \mathbf{E}[G_\theta],$$

hence we have proved the result.

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